Determinants

Every square matrix has a unique real number associated to it, called a determinant.

Consider the $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{m2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{ij} \end{bmatrix}$$

Definition

Determinant of a 2×2 matrix.

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

Example

$$\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1(4) - 2(3) = -2 \qquad \det \begin{bmatrix} 5 & 6 \\ -10 & -8 \end{bmatrix} = 5(-8) - 6(-10) = 20$$

Definition

The **minor** of the entry a_{ij} denoted by M_{ij}, is the determinant of the submatrix after removing the i^{th} row and j^{th} column of the matrix A.

Example

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 0 & 3 & 1 \\ -1 & 6 & 2 \end{bmatrix}$$

$$M_{11} = \det \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix} = 3(2) - 1(6) = 0 \qquad M_{31} = \det \begin{bmatrix} 1 & 5 \\ 3 & 1 \end{bmatrix} = 1(1) - 5(3) = -14$$

$$M_{32} = \det \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} = 1(1) - 5(0) = 1$$

Definition

The **cofactor** of the entry a_{ij} of a matrix A, denoted by C_{ij}, is given by

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Going back to the previous example,

$$C_{11} = (-1)^{1+1} M_{11} = M_{11} = 0$$

$$C_{31} = (-1)^{3+1} M_{31} = -14$$

$$C_{32} = (-1)^{3+2} M_{32} = -M_{32} = -1$$

Determinants using cofactor expansion

If an $n \times n$ matrix A is defined as previously, then

$$\det A = \sum_{j=1}^{n} a_{ij} \cdot C_{ij} \qquad \text{(cofactor expansion about the } i^{th} \text{ row)}$$

$$\det A = \sum_{i=1}^{n} a_{ij} \cdot C_{ij} \qquad \text{(cofactor expansion about the } j^{th} \text{ column)}$$

Examples

1. a. $\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = a_{11} \cdot C_{11} + a_{12} \cdot C_{12} + a_{13} \cdot C_{13}$ (cofactor expansion about 1st row)

$$= 1(-1)^{1+1} \det \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} + 2(-1)^{1+2} \det \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} + 3(-1)^{1+3} \det \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$$

$$= 1(45-48) - 2(36-42) + 3(32-35) = -3 + 12 - 9 = 0$$

b. $\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = a_{13} \cdot C_{13} + a_{23} \cdot C_{23} + a_{33} \cdot C_{33}$ (cofactor expansion about 3rd column)

$$= 3(-1)^{1+3} \det \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix} + 6(-1)^{2+3} \det \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix} + 9(-1)^{3+3} \det \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

$$= 3(32-35) - 6(8-14) + 9(5-8) = -9 + 36 - 27 = 0$$

2. a. $\det \begin{bmatrix} 1 & 2 & 0 \\ -2 & 2 & 0 \\ 1 & 4 & 3 \end{bmatrix} = 1 \cdot C_{11} + 2 \cdot C_{12} + 0 \cdot C_{13}$

$$= 1 \cdot \det \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} - 2 \cdot \det \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} = 6 - 2 \cdot (-6) = 18$$

b. $\det \begin{bmatrix} 1 & 2 & 0 \\ -2 & 2 & 0 \\ 1 & 4 & 3 \end{bmatrix} = 0 \cdot C_{13} + 0 \cdot C_{23} + 3 \cdot C_{33}$

$$= 3 \cdot \det \begin{bmatrix} 1 & 2 \\ -2 & 2 \end{bmatrix} = 3(2+4) = 18$$

Properties of Determinants

1. If a matrix has a row of zeros, then its determinant is zero.
2. For any square matrices A and B, $\det(AB) = \det A \cdot \det B$.
3. If two rows of matrix A are interchanged to produce matrix B, $\det B = -\det A$.
4. When matrix B is obtained from A by multiplying a row of A by a nonzero constant c, $\det B = c \det A$.
5. When matrix B is obtained by adding a multiple of a row of A to another row of A, $\det B = \det A$.

6. $$\det\begin{bmatrix} a_{11} & * & \cdots & * \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix} = \det\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ * & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \cdots & * & a_{nn} \end{bmatrix} = a_{11}\cdot a_{22}\cdot a_{33}\cdots a_{nn}$$

Properties of determinants in action

$$\det\begin{bmatrix} 1 & 2 & 0 \\ -2 & 2 & 5 \\ 1 & 8 & 3 \end{bmatrix} = \det\begin{bmatrix} 1 & 2 & 0 \\ 0 & 6 & 5 \\ 0 & 6 & 3 \end{bmatrix} = \det\begin{bmatrix} 1 & 2 & 0 \\ 0 & 6 & 5 \\ 0 & 0 & -2 \end{bmatrix} = 1\cdot(6)\cdot\left(-2\right) = -12$$

$$\det\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 1 & 5 \\ -2 & 1 & 0 & 4 \\ 0 & 2 & 0 & 0 \end{bmatrix} = \det\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 1 & 5 \\ 0 & 1 & 6 & 4 \\ 0 & 2 & 0 & 0 \end{bmatrix} = -\det\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 6 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 2 & 0 & 0 \end{bmatrix} = -\det\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 6 & 4 \\ 0 & 0 & -11 & -3 \\ 0 & 0 & -12 & -8 \end{bmatrix}$$

$$= 11\cdot\det\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 6 & 4 \\ 0 & 0 & 1 & 3/11 \\ 0 & 0 & -12 & -8 \end{bmatrix} = 11\cdot\det\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 6 & 4 \\ 0 & 0 & 1 & 3/11 \\ 0 & 0 & 0 & -52/11 \end{bmatrix} = -52$$

Theorem 4

If an $n\times n$ matrix A is invertible (it has an inverse),

$$\det A = \frac{1}{\det A^{-1}}$$

Proof

$$AA^{-1} = I_n \qquad \text{(apply determinants on both sides)}$$

$$\det\ AA^{-1} = \det I_n$$

$$\det A\cdot\det A^{-1} = 1 \qquad \text{(apply properties of determinants)}$$

$$\det A = \frac{1}{\det A^{-1}}$$

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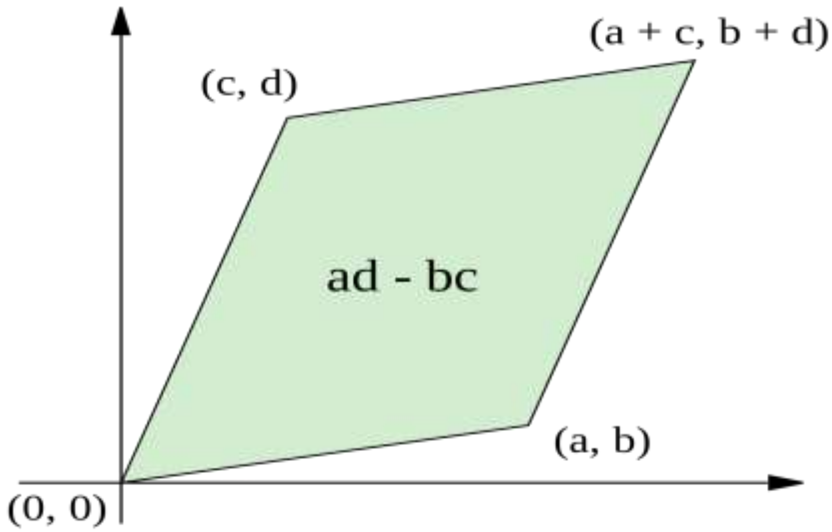
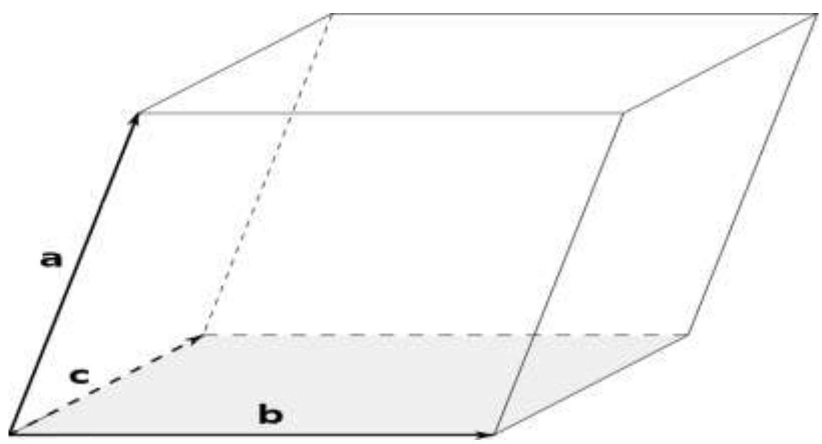
Geometry of Determinants



Let $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. The **area** of the parallelogram formed by the columns of A is precisely $|\det A|$.



Let $A = \begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$. The **volume** of the parallelepiped formed by the columns of A is precisely $|\det A|$.

Homework

1. Find the determinant of $A = \begin{bmatrix} -3 & 6 & 9 \\ 9 & 12 & -3 \\ 0 & 15 & -6 \end{bmatrix}$ using

   a. cofactor expansion about the 2nd column;
   b. cofactor expansion about the 2nd row;
   c. properties of determinants;

2. Find the determinant of

$$C = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 1 & 5 \\ -2 & 1 & 0 & 4 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

3. **True** or **False**

a. _____ For any matrices A and B $\det(A+B) = \det(A) + \det(B)$

b. _____ For any matrix A and any constant c, $\det(c \cdot A) = c \cdot \det(A)$

c. _____ If a matrix has a row of zeros, then its determinant must be zero.

d. _____ If $\det A = 0$, the matrix A has no inverse.